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PACKING DIMENSION ESTIMATION FOR EXCEPTIONAL PARAMETERS

BY

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ABSTRACT

We estimate by a combinatorial method the packing dimension of the set of parameters for which the limit set of an iterated function system has a drop of the Hausdorff dimension.

1. Introduction

Let V be an open and bounded subset of $\mathbb{R}^d$. For each parameter $t \in \bar{V}$ we consider a conformal iterated function system (IFS) $(f_i(\cdot,t))_{i=1}^k$ in $\mathbb{R}^d$ depending on the parameter t. We assume this dependence to be smooth (at least $C^{1+\beta}$). We denote by Λ_t the limit set of the IFS. We also denote by $s(t)$ the solution of the Bowen's equation

$$P(s(t)\chi_t) = 0$$

where χ_t is the Lyapunov exponent of the IFS and P denotes the topological pressure. It is well known that

$$\dim_H \Lambda_t \le s(t).$$

If, in addition, the IFS satisfies the Open Set Condition (OSC), it is a classical result (due to Hutchinson [Hu] and Manning and McCluskey [MM]) that the

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Hausdorff dimension of the limit set equals $s(t)$. In this paper we do not assume the OSC is satisfied. Without this assumption we study the set of parameters t for which the Hausdorff dimension of the limit set is strictly smaller than $s(t)$. In particular, assuming Policott and Simon's transversality condition, we are interested in estimating the dimension of the set of parameter values t for which

$$\dim_H \Lambda_t \leq p$$

for any p.

THEOREM 1.1: *For each* p, *let*

$$G_p = \{t \in \bar{V} \colon \dim_H \Lambda_t \leq p\}.$$

If the IFS $(f_i(\cdot,t))_{i=1}^k$ *satisfies the transversality condition (Definition 4.4) and* $p < \min(d, s(u))$ *then*

$$\limsup_{r \searrow 0} \dim_P(G_p \cap B_r(u)) \leq p,$$

where $\dim_P$ *denotes the packing dimension.*

We refer the reader to the paper [PS] for both the estimations of the 'exceptional' parameters set in a much more general setting and the historical information on the problem.

The paper is organized as follows. In the second section we introduce the notation we use and give basic information on IFS. The third section contains definitions and properties of intersection numbers. In the fourth section we give some technical results in parameter space. Finally, the fifth section contains the last steps of the proof of Theorem 1.1.

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2. Background

Two words about notation first. Symbols c, c_2, $\tilde{c}$ and the like denote internal constants, to be used only within one proof. The same symbols in different lemmas and propositions do not necessarily denote the same constant. When the constant from one statement is to be used later, it will be denoted differently.

Symbol $\approx$ means equality up to a bounded multiplicative constant. Symbol $|\cdot|$ means diameter (when used for sets). For a conformal mapping f from $\mathbb{R}^d$ into itself we will denote $f' = |\det Df|^{1/d}$ — this is called the local contraction (dilatation) ratio.

Iterated function system is a finite family $(f_i)_{i=1}^k$ of contractive diffeomorphisms acting from $\mathbb{R}^d$ into itself. The **limit set** of IFS is the unique non-empty compact set Λ satisfying the equation

$$\Lambda = \bigcup f_i(\Lambda).$$

The **symbolic space** of an IFS is defined as

$$\Sigma = \{1, \ldots, k\}^{\mathbb{N}};$$

its elements will be denoted by $\omega = (\omega_1\omega_2\ldots)$. The finite sequences of symbols $1, \ldots, k$ will be denoted by $\omega^n = (\omega_1\omega_2\ldots\omega_n)$. The set of all such finite sequences will be denoted $\Sigma_0 = \bigcup_{i=0}^{\infty}\{1, \ldots, k\}^i$.

On Σ we define two mappings. The **left shift** σ deletes the first digit of a sequence:

$$\sigma(\omega_1\omega_2\ldots) = (\omega_2\omega_3\ldots).$$

The **right shift** σ_i $(i = 1, 2, \ldots, k)$ adds the symbol i at the beginning of the sequence:

$$\sigma_i(\omega_1\omega_2\ldots) = (i\omega_1\omega_2\ldots).$$

We write $f_{\omega^n} = f_{\omega_1} \circ \cdots \circ f_{\omega_n}$. We define a projection from Σ into $\mathbb{R}^d$:

$$\pi(\omega) = \lim_{n\to\infty} f_{\omega^n}(0). \tag{2.1}$$

By [Hu], $\Lambda = \pi(\Sigma)$. When $x = \pi(\omega)$, ω will be called the **symbolic expansion** of x (it need not be uniquely defined). Dynamics on Λ (given by f_i's) is a factor of the dynamics on Σ (given by right shifts σ_i's):

$$f_i \circ \pi = \pi \circ \sigma_i.$$

We demand that all the mappings f_i are smooth (at least $C^{1+\beta}$) and conformal, at least in some neighbourhood of Λ. The latter assumption is void in the one-dimensional case but quite restricting (allowing only Möbius transformations) when $d \geq 3$. We denote by U the neighbourhood of Λ on which these assumptions are satisfied and assume U is bounded, open, simply connected.

As f_i are contractions with universally bounded contraction ratio, also f_ω^n satisfy the following inequality (called Bounded Distortion Property, BDP):

$$\exists L_0 \forall_{x,y,z\in U} \forall_{\omega^n \in \Sigma_0} \quad L_0^{-1} \leq \frac{|f_{\omega^n}(x) - f_{\omega^n}(y)|}{|x-y| f'_{\omega^n}(z)} \leq L_0; \tag{2.2}$$

see [F2], chapter 4.

As U is an open neighbourhood of the compact set Λ, there exist two constants L_1, L_2 such that

$$B_{L_1|U|}(\Lambda) \subset U \subset B_{L_2|U|}(\Lambda). \tag{2.3}$$

Because of (2.2), the family $U_{\omega^n} = f_{\omega^n}(U)$ is **regular**, that is the ratio of the outer radius (the radius of the smallest ball containing the set) and the inner radius (the radius of the greatest ball contained in the set) is uniformly bounded in the family. We can freely demand that $U_i \subset U$ for all i (for example, $U = B_r(\Lambda)$ will satisfy this). It follows that $U_{\omega^n i} \subset U_{\omega^n}$. The sets U_{ω^n} will be called **cylinders**.

We denote by $\lambda^+_{\omega^n}$ and $\lambda^-_{\omega^n}$ the maximum and minimum of the local contraction ratio of f_{ω^n} in U. We also denote $\lambda^+ = \max_i \lambda^+_i$ and $\lambda^- = \min_i \lambda^-_i$ (i.e., the maximum and minimum of local contraction ratios of all the mapping f_i).

We will use sets $\Sigma_{\omega^n} = \sigma_{\omega^n}(\Sigma) = \sigma_{\omega_1} \circ \cdots \circ \sigma_{\omega_n}(\Sigma)$ and call them cylinders, too (in symbolic space). It is easy to see that $\pi(\Sigma_{\omega^n}) \subset U_{\omega^n}$; the cylinders U_{ω^n} and Σ_{ω^n} will be called **dual**.

We introduce a metric on Σ given by $\rho(\omega, \tau) = |U_{\omega^n}|$, where $\omega_i = \tau_i$ for all $i \leq n$ but not for $i = n+1$. If $\omega_1 \neq \tau_1$, then $\rho(\omega,\tau) = |U|$. This metric agrees with the product topology on Σ. It is easy to check that π is a Lipschitz mapping (with Lipschitz constant 1) in the metric ρ.

We say that a family of cylinders $Z_l = \{U_{\omega^n} : \omega^n \in T \subset \Sigma_0\}$ forms a **Moran cover** if their diameters are between $v^{-1}l$ and vl (for some v) and their dual cylinders are disjoint and cover Σ. The minimal such v is called the **variation** of Moran cover Z_l. The following is standard.

LEMMA 2.1: *Take $l \leq |U|$. Let v be the maximal ratio of diameters of a cylinder and its immediate subcylinder. We can construct a Moran cover of size l and variation not greater than $v^{1/2}$.*

Proof: Note first that the ratio $|U|/|r(U_j)|$ is bounded (in j), where r denotes the inner radius. Hence (by BDP (2.2)) also the ratio $|U_{\omega^n}|/|U_{\omega^n j}|$ is uniformly bounded. We denote its upper bound by v.

We will construct the Moran cover by a repeated subdivision procedure: we start from $T_0 = \{\emptyset\}$. Then for all $\omega^n \in T_i$ we check if $|U_{\omega^n}| \leq v^{1/2}l$. If yes, we put ω^n in T_{i+1}. If no, we put all the sequences $\omega^n j$ $(j = 1, \ldots, k)$ in T_{i+1}. On each step $\{\Sigma_{\omega^n} : \omega^n \in T_i\}$ is a family of disjoint cylinders, covering Σ. Because all the mappings f_i are contractions with contraction ratios bounded away from 1, the diameters of cylinders U_{ω^n} vanish exponentially fast with n, hence the procedure must stop (i.e., $T_i = T_{i+1} = \cdots$ for some i). All the cylinders $U_{\omega^n} : \omega^n \in T_i$ have diameters between $v^{-1/2}l$ and $v^{1/2}l$, hence they form a Moran cover we were looking for. ∎

By the **Moran covering** we will mean a choice of Moran covers for all $l \leq |U|$ with uniformly bounded variations. As follows from the Lemma 2.1, we can always construct a Moran covering of not too small variation.

We will use now some facts from the thermodynamical formalism; see, for example, [Bo]. We define a family of Hölder continuous functionals on Σ:

$$\phi_r(\omega) = r \cdot \log(f'_{\omega_1} \circ \pi \circ \sigma(\omega)).$$

There exists precisely one value $r = s$ for which the pressure of ϕ_r vanishes; we call it the **similarity dimension** of IFS. Let μ be the Gibbs measure (on Σ) for ϕ_s and denote by ν the projection of μ under π. We will call ν the **natural measure** of the IFS.

From the definition of Gibbs measure

$$\mu(\Sigma_{\omega^n}) \approx \exp\Bigg(\sum_{i=0}^{n-1} \phi_s(\sigma^{\circ i}(\omega))\Bigg) \approx \Big(\frac{|U_{\omega^n}|}{|U|}\Big)^s. \tag{2.4}$$

Hence, μ is equivalent to the s-dimensional Hausdorff measure on (Σ, ρ). In particular, the Hausdorff dimension of Λ cannot be greater than s (as the Lipschitz mappings cannot increase the Hausdorff dimension) and the Moran cover of size l has approximately l^{-s} elements, up to a multiplicative constant depending on its variation.

For a family of IFS, the sequences giving a Moran covering for one parameter value will not give in general a Moran covering for different parameter values. We will use a less restrictive notion. Given a family of Moran covers (Z_l) we will call it D**-covering** if the variation of the Moran cover Z_l is not greater than $v \cdot (l/|U|)^D$, hence not necessarily uniformly bounded in the whole family. We will use the notion for small D only.

3. Intersection numbers

Let us choose one Moran covering $\{Z_l \colon l \leq |U|\}$ with variation L_3. By (2.4), the dual cylinders to those in Z_l have measure μ between $L_4^{-1}l^s$ and $L_4 l^s$ for some L_4 for all l. Hence Z_l has between $L_4^{-1}l^{-s}$ and $L_4 l^{-s}$ elements. Furthermore, if a cylinder belongs to Z_l, then no less than $L_4^{-2}(\tilde{l}/l)^{-s}$ and no more than $L_4^2(\tilde{l}/l)^{-s}$ of its subcylinders belong to $Z_{\tilde{l}}$, $\tilde{l} < l$.

PROPOSITION 3.1: *The correlation dimension of the natural measure of a conformal IFS exists.*

Proof: The statement very similar to our assertion is proved in [PeSo]. The proof easily translates to our setting.

Let D_a be the partition of $\mathbb{R}^d$ into dyadic grid boxes of size 2^{-a}. The correlation dimension ([Pe]) is defined by

$$\dim_C(\nu) = \lim_{n\to\infty} \frac{\log \tau_n^{(2)}(\nu)}{-n\log 2},$$

where $\tau_n^{(2)} = \sum_{Q\in D_n}(\nu(Q))^2$. Denote

$$p_{\omega^n}^{+} = \sup_{\kappa^m} \frac{\mu(\Sigma_{\omega^n\kappa^m})}{\mu(\Sigma_{\kappa^m})}$$

and

$$p_{\omega^n}^{-} = \inf_{\kappa^m} \frac{\mu(\Sigma_{\omega^n\kappa^m})}{\mu(\Sigma_{\kappa^m})}.$$

Let $P \in D_{a+b}$, $Q \in D_b$, $P \subset Q$. We have

$$\nu(P) \leq \sum_{\omega^n\in Z_l \colon U_{\omega^n}\cap Q\neq\emptyset} p_{\omega^n}^{+}\nu(f_{\omega^n}^{-1}(P)). \tag{3.1}$$

Denote

$$p_+(Q) = \sum_{\omega^n\in Z_l \colon U_{\omega^n}\cap Q\neq\emptyset} p_{\omega^n}^{+}$$

and $p_-(Q)$ analogously. Now we will follow the proof of theorem 1.1 in [PeSo]. It is enough to prove that

$$\tau_{a+b}^{(2)}(\nu) \leq c\tau_a^{(2)}(\nu)\tau_b^{(2)}(\nu);$$

then the result will follow by a sub-multiplicative argument. The same calculations as in [PeSo], using (3.1), give us

$$\tau_{a+b}^{(2)}(\nu) \leq c_1\tau_a^{(2)}(\nu)\sum_{Q\in D_b} p_+(Q)^2$$

and

$$\sum_{Q\in D_b} p_-(Q)^2 \le c_2\tau_b^{(2)}(\nu).$$

By (2.4), the ratio $p^+_{\omega^n}/p^-_{\omega^n}$ is uniformly bounded for all ω^n. The result follows. ∎

We define the **intersection number**

$$\tilde{A}_l = \sharp\{(U_{\omega^n}, U_{\kappa^m}) \in Z_l \times Z_l \colon U_{\omega^n} \cap U_{\kappa^m} \neq \emptyset\}$$

(note that we do not exclude the case $\omega^n = \kappa^m$). It lies somewhere between $\sharp Z_l \approx l^{-s}$ and $\sharp(Z_l \times Z_l) \approx l^{-2s}$.

PROPOSITION 3.2: *The limit*

$$\tilde{s} = \lim_{l\searrow 0} \frac{\log \tilde{A}_l}{-\log l}$$

exists and equals $2s - \dim_C(\nu)$. *It does not depend on the choice of* U *and* Z_l.

Proof: We use another, equivalent definition of the correlation dimension:

$$\dim_C(\nu) = \lim_{r\searrow 0} \frac{1}{\log r} \log \int \nu(B_r(x))d\nu(x).$$

The pair (U, Λ) is mapped by any f_{ω^n} in a conformal way with uniformly bounded distortion, hence (by (2.2))

$$B_{L_0^{-1}L_1|U_{\omega^n}|}(\pi(\Sigma_{\omega^n})) \subset U_{\omega^n} \subset B_{L_0L_2|U_{\omega^n}|}(\pi(\Sigma_{\omega^n}))$$

for all ω^n. Hence, given any $x \in \Lambda$, any $l \le |U|$ and any $U_{\omega^n} \in Z_l$, $x \in \pi(\Sigma_{\omega^n})$, we have $B_{L_0^{-1}L_1L_3^{-1}l}(x) \subset U_{\omega^n}$ (because of the uniform variation of the Moran covering). For the same reason the ball $B_{(L_0L_2+1)L_3l}(x)$ contains all the cylinders from Z_l, intersecting U_{ω^n}.

Let $N(U_{\omega^n})$ be the number of cylinders from Z_l, intersecting U_{ω^n} (U_{ω^n} itself included). As their dual cylinders have measure μ approximately equal to l^s, each

$$\nu(B_{L_0^{-1}L_1L_3^{-1}l}(x)) \le L_4 l^s N(U_{\omega^n})$$

and

$$\nu(B_{(L_0L_2+1)L_3l}(x)) \ge L_4^{-1} l^s N(U_{\omega^n}).$$

We sum these inequalities over all $U_{\omega^n} \in Z_l$, using the fact that their measure is approximately equal to l^s, again. We get

$$\int \nu(B_{L_0^{-1}L_1L_3^{-1}l}(x))d\nu(x) \leq L_4^2 l^{2s} \sum_{U_{\omega^n}\in Z_l} N(U_{\omega^n})$$

and

$$\int \nu(B_{(L_0L_2+1)L_3l}(x))d\nu(x) \geq L_4^{-2} l^{2s} \sum_{U_{\omega^n}\in Z_l} N(U_{\omega^n}).$$

Using now the definition $\tilde{A}_l = \sum N(U_{\omega^n})$, we get

$$L_4^{-2} l^{2s} \tilde{A}_{L_0L_1^{-1}L_3r} \leq \int \nu(B_r(x))d\nu(x) \leq L_4^2 l^{2s} \tilde{A}_{L_3^{-1}r/(L_0L_2+1)}$$

for all r. Now the assertion follows from the existence of correlation dimension. ∎

An additional corollary follows from the proof above — that when we change U, all the numbers $\tilde{A}_l$ will change at most by a constant depending on U.

As mentioned earlier, for a family of IFS we will need to work with D-coverings instead of Moran coverings. Let (Z_l^D) be a D-covering, with variation of the Z_l^D Moran cover not greater than $L_3(l/|U|)^{-D}$. We define $\tilde{A}_l^D$ in the same way that $\tilde{A}_l$ was defined.

We first notice that by a repeated subdivision procedure we can change the D-covering into a Moran covering. The cylinders of Z_l^D have diameters between $L_3^{-1}l^{1+D}|U|^{-D}$ and $L_3 l^{1-D}|U|^D$. We divide the cylinders until their diameters are between $L_3^{-1}l^{1+D}|U|^{-D}$ and $L_3 l^{1+D}|U|^{-D}$. We get a Moran cover of variation L_3 and size $l^{1+D}|U|^{-D}$. These new Moran covers form a Moran covering. Note that all cylinders from Z_l^D were divided onto subcylinders of diameter no more than $L_3^2(l/|U|)^{-2D}$ times smaller, hence at most $L_4L_3^{2s}(l/|U|)^{-2Ds}$ of them.

For every pair of intersecting cylinders from Z_l^D we will thus get at most $L_4^2L_3^{4s}(l/|U|)^{-4Ds}$ intersecting pairs in $Z_{l^{1+D}|U|^{-D}}$. Hence

$$\tilde{A}_{l^{1+D}|U|^{-D}} \leq L_4^2L_3^{4s}\left(\frac{l}{|U|}\right)^{-4sD}\tilde{A}_l^D. \tag{3.2}$$

This (together with Proposition 3.2) implies

$$\dim_C(\nu) \geq \frac{2s(1-D)}{1+D} - \frac{1}{1+D}\liminf_{l\searrow 0}\frac{\log \tilde{A}_l^D}{-\log l}. \tag{3.3}$$

We are basically interested in the Hausdorff dimension of the limit set. However, as follows from the capacity definition of Hausdorff dimension (see [F1]), the correlation dimension of ν is a lower bound for the Hausdorff dimension of Λ. Hence the set of parameter values for which the Hausdorff dimension of the limit set drops is contained in the set of parameter values for which the correlation dimension of the natural measure drops and this is the set we will work with in the last section.

4. Transversality

We will now consider not a single IFS but a d-dimensional family of IFS, acting on $\mathbb{R}^d$. We will use $t = (t_1, \ldots, t_d)$ as a parameter and write the dependence on t explicitly; for example, the limit set will be denoted as Λ_t. The set of parameters $\bar{V}$ is assumed to be the closure of a bounded open subset of $\mathbb{R}^d$. We assume the contractions $f_i(x;t)$ to be $C^{1+\beta}$ for $(x,t) \in U \times \bar{V}$, that is we want all the derivatives $\partial f_i/\partial x_j$ and $\partial f_i/\partial t_j$ to be C^β with respect to both x and t.

Let us start with two technical lemmas.

LEMMA 4.1: *Given ω, $\pi_t(\omega)$ is $C^{1+\beta}$ as a function of t. The Hölder constant for $\partial\pi_t(\omega)/\partial t_j$ can be chosen independently of ω.*

Proof: We use the definition (2.1) of π_t. The limit in (2.1) is approached at an exponential rate, hence the derivative of the limit equals the limit of derivatives. We use the formula

$$\frac{\partial}{\partial t}(f_1 \circ \ldots \circ f_n)(x) = \sum_{i=1}^{n} D_x(f_1 \circ \ldots \circ f_{i-1})(f_i \circ \ldots \circ f_n(x)) \cdot \frac{\partial}{\partial t} f_i(f_{i+1} \circ \ldots \circ f_n(x))$$

and get

$$H_j(t) := \frac{\partial}{\partial t_j}\pi_t(\omega) = \sum_{i=0}^{\infty} D_x f_{\omega^i}(\pi_t(\sigma^i\omega)) \cdot \frac{\partial}{\partial t_j} f_{\omega_{i+1}}(\pi_t(\sigma^{i+1}\omega)).$$

We remind the reader that ω^i is the sequence of the first i symbols while ω_{i+1} is the $(i+1)$-st symbol of the sequence ω. Also, σ^i is the i-th power of the left shift σ, not the i-th right shift which is denoted σ_i.

First we prove H_j to be bounded in t. It is so because $|D_x f_{\omega^i}|$ decays exponentially fast with i and $\partial/\partial t_j(f_{\omega_i})$ is bounded.

Now we want to prove that H_j is Hölder. Take $t, u \in \bar{V}$. We have

$$D_x f_i(\pi_t(\omega);t) - D_x f_i(\pi_u(\omega);u) =$$

$$[D_x f_i(\pi_t(\omega);t) - D_x f_i(\pi_u(\omega);t)] + [D_x f_i(\pi_u(\omega);t) - D_x f_i(\pi_u(\omega);u)]. \tag{4.1}$$

In the first bracket at the right hand side of (4.1) we have the difference of values of the C^β function, computed at two points at a distance not greater than $c_0|t-u|$ (from boundedness of H_j). In the second bracket we have the difference of values of the C^β function, computed at two points lying at distance $|t-u|$. All in all, the sum is not greater than $c_1|t-u|^\beta$. The constant c_1 can be chosen independently of ω and i by a compactness argument. Hence

$$|H_j(t)-H_j(u)| \le \left(c_1 \sum_i |D_x f_{\omega^i}|\right)|t-u|^\beta \le c_2|t-u|^\beta$$

and we are done. ∎

Lemma 4.2: *Given $u \in \tilde{V}$, $\omega^n \in \Sigma_0$ and $x \in U$, the function*

$$F_{x,\omega^n}(t) = \frac{\log f'_{\omega^n}(x;t)}{\log f'_{\omega^n}(x;u)}$$

is C^β as a function of t, with Hölder constant independent of x and ω^n.

Proof: We recall the notation $f' = (\det D_x f)^{1/d}$. We have to prove that

$$|\log f'_{\omega^n}(x;t) - \log f'_{\omega^n}(x;u)| \le c|t-u|^\beta |\log f'_{\omega^n}(x;u)|.$$

Denote

$$g_{\omega^n}(x;t) = |d \log f'_{\omega^n}(x;t)| = |\log \det D_x f_{\omega^n}(x;t)|.$$

As f'_i is C^β and bounded away from 0, also its logarithm has to be C^β. Using the formula

$$(f_1 \circ \ldots \circ f_n)'(x) = \prod_{i=1}^n f'_i(f_{i+1} \circ \ldots \circ f_n(x))$$

we get

$$|g_{\omega^n}(x;t) - g_{\omega^n}(x;u)| \le \sum_{i=1}^n |g_{\omega_i}(f_{\sigma^i\omega^n}(x;t),t) - g_{\omega_i}(f_{\sigma^i\omega^n}(x;u),u)|.$$

Like in Lemma 4.1, $f_{\sigma^i\omega^n}(x;t)$ is $C^{1+\beta}$ with respect to t; in particular it is Lipschitz. Writing the expression under the sum in the form

$$|[g_{\omega_i}(f_{\sigma^i\omega^n}(x;t),t) - g_{\omega_i}(f_{\sigma^i\omega^n}(x;u),t)] + [g_{\omega_i}(f_{\sigma^i\omega^n}(x;u),t) - g_{\omega_i}(f_{\sigma^i\omega^n}(x;u),u)]|$$

we can estimate it by $c_1|t-u|^\beta$, like in the estimation of the right hand side of formula (4.1). The following inequality lets us estimate n from above (i.e. relate it to g_{ω^n}):

$$g_{\omega^n}(x;u) \ge n|\log \lambda^+(u)|.$$

Hence

$$|g_{\omega^n}(x;t) - g_{\omega^n}(x;u)| \le \frac{c_1 g_{\omega^n}(x;u)}{|\log \lambda^+(u)|}|t-u|^\beta$$

and the assertion follows. ∎

COROLLARY 4.3: *Similarity dimension $s(t)$ is a C^β function of t.*

Proof: We recall the definition of pressure:

$$P(r;t) = \lim_{n\to\infty} \frac{1}{n} \log M_n^-(r;t) = \lim_{n\to\infty} \frac{1}{n} \log M_n^+(r;t),$$

where

$$M_n^\pm(r;t) = \sum_{\omega^n \in \{1,\ldots,k\}^n} (\lambda_{\omega^n}^\pm(t))^r.$$

Given $u \in \bar{V}$, both

$$\frac{\log \lambda_{\omega^n}^-(t)}{\log \lambda_{\omega^n}^-(u)}$$

and

$$\frac{\log \lambda_{\omega^n}^+(t)}{\log \lambda_{\omega^n}^+(u)}$$

are C^β functions of t, with Hölder constant independent of ω^n — this follows easily from Lemma 4.2 applied to two points in U: one where $\lambda_{\omega^n}^-(t)$ is achieved and one where $\lambda_{\omega^n}^-(u)$ is achieved (respectively, $\lambda_{\omega^n}^+(t)$ and $\lambda_{\omega^n}^+(u)$).

Hence, given r, the pressure is a Hölder function of t. As its derivative is bounded from below, the zero of the pressure function (= similarity dimension) also depends on t in a Hölder way. ∎

Denote $h_{\omega,\kappa}(t) = \pi_t(\omega) - \pi_t(\kappa)$. The following definition was first introduced (in a one-dimensional situation) in [PoSi].

Definition 4.4: The family of IFS satisfies the **transversality condition** if there exists a constant L_5 such that for any two sequences $\omega, \kappa \in \Sigma$ with $\omega_1 \ne \kappa_1$, if $|h_{\omega,\kappa}(u)| < L_5$ then $|\det D_t h_{\omega,\kappa}(t)|_{|t=u} > L_5$.

Given two sequences $\omega^n, \kappa^m \in \Sigma_0$ we denote by $I(\omega^n, \kappa^m)$ the set of parameters t for which the cylinders $U_{\omega^n}(t)$ and $U_{\kappa^m}(t)$ intersect each other. We will try to describe these sets using transversality.

LEMMA 4.5: *Assume the transversality condition is satisfied; ω and κ are like in its definition. Then any one component of $(h_{\omega,\kappa})^{-1}(B_r(0))$, $r < L_5$, is contained in some ball of radius $L_6 r$, L_6 not depending on ω or κ.*

Proof: The directional derivatives (over t_j) of $h_{\omega,\kappa}$ are bounded from above by compactness of $\bar{V}$. The determinant of $D_t h_{\omega,\kappa}$ is bounded from below on $(h_{\omega,\kappa})^{-1}(B_r(0))$ by transversality. Hence each branch of $(h_{\omega,\kappa})^{-1}$ is locally well defined on $B_r(0)$ and its norm is uniformly bounded from above. ∎

PROPOSITION 4.6: *Assume the transversality condition is satisfied. Choose $u \in \bar{V}$ and let $(Z_l(u))$ be a Moran covering in u. Then for any small v there exists an open set $W \ni u$ and a constant $Y(v)$ such that for any $l < |U|$ and for any two cylinders $U_{\omega^n}(u), U_{\kappa^m}(u) \in Z_l(u)$, $\omega_1 \neq \kappa_1$, the set $W \cap I(\omega^n, \kappa^m)$ can be covered with at most Y balls of radius Yl^{1-v} each.*

Proof: We need to prove the assertion for small l only; for big l it is automatically satisfied. Take small l; let ω^n and κ^m be like in the assertion. Let ω and κ be any two sequences, beginning with ω^n and κ^m, respectively. We denote $x(t) = \pi_t(\omega)$, $y(t) = \pi_t(\kappa)$ and $H(t) = |x(t) - y(t)|$. Choose v. The cylinders $U_{\omega^n}(u)$ and $U_{\kappa^m}(u)$ have diameters approximately equal to l. Hence, from Lemma 4.2 for W small enough,

$$\max(|U_{\omega^n}(t)|, |U_{\kappa^m}(t)|) < l^{1-v}$$

for all $t \in W$. We may assume l is so small that $l^{1-v} < L_5/2$, where L_5 is the constant from the definition of transversality condition. We see that

$$U_{\omega^n}(t) \subset B_{l^{1-v}}(x(t))$$

and

$$U_{\kappa^m}(t) \subset B_{l^{1-v}}(y(t)).$$

When $t \in I(\omega^n, \kappa^m)$, the points $x(t)$ and $y(t)$ have to be at a distance not greater than L_5. By Lemma 4.5 all the components of $\{t\colon H(t) \leq 2l^{1-v}\} \supset I(\omega^n, \kappa^m) \cap W$ are of diameter at most $L_6 l^{1-v}$.

We have only to estimate the number of such components. Let W_1 and W_2 be two of them. We choose the closest to every other point of W_1 and W_2 (denote them $t^{(1)}$ and $t^{(2)}$) and draw a line segment α between them. If the closest two points are not uniquely defined, we choose one of the pairs at minimal distance. At these points $H(t^{(i)}) = 2l^{1-v}$.

We consider the directional derivative of $H(t)$ along α. From transversality, like in the proof of Lemma 4.5, at the points $t^{(1)}$ and $t^{(2)}$ it is bounded away from zero (positive in $t^{(1)}$, negative in $t^{(2)}$, of order at least c/L_6 at both points). But by Lemma 4.1 H is in class $C^{1+\beta}$, hence its directional derivative is Hölder. The value of a Hölder function cannot change too much in too short a distance, hence the length of α is uniformly bounded from below.

All the components of $\{t\colon H(t) \le 2l^{1-v}\} \cap W$ are thus at a bounded-from-below distance from each other and hence their number depends only on the size of W and not on l. We are done. ∎

5. Parameter space

We want to estimate the size of the set of parameters for which the condition on growth of intersection numbers given in Proposition 3.2 is fulfilled. To do that we will first rewrite them using another kind of intersection number. We define

$$A_l = \sharp\{(U_{\omega^n}, U_{\kappa^m}) \in Z_l \times Z_l\colon U_{\omega^n} \cap U_{\kappa^m} \neq \emptyset, \omega_1 \neq \kappa_1\}.$$

Note that the inequality (3.2) stays true for these new intersection numbers as well (with the same proof).

LEMMA 5.1: *If $\tilde{l} < l$, then*

$$A_{\tilde{l}} \le L_4^4(\tilde{l}/l)^{-2s} A_l.$$

Proof: The proof is similar to that of the inequality (3.2). A cylinder from Z_l has at most $L_4^2(\tilde{l}/l)^{-s}$ subcylinders in $Z_{\tilde{l}}$. If two cylinders in Z_l are disjoint, their subcylinders are disjoint, too. If two cylinders in Z_l are not disjoint, there can be at most $L_4^4(\tilde{l}/l)^{-2s}$ intersections among their subcylinders. ∎

To write the dependence of A_l and $\tilde{A}_l$, we will assume (from now on) that the Moran coverings we use are chosen in a special way. Namely, we demand that there exists a sequence $l_i \searrow 0$ such that every cylinder U_{ω^n} belongs to at least one and at most some L_7 of Moran covers Z_{l_i}. For L_7 big enough it is easy to satisfy.

LEMMA 5.2: *For some L and $\gamma < 1$ for any $j > i$*

$$l_j \le L\gamma^{j-i}.$$

Proof: Choose any $\omega \in \Sigma$ and $U_{\tau^n} \in Z_{l_i}$. The sequence

$$\{|U_{\tau^n}|, |U_{\tau^n\omega^1}|, |U_{\tau^n\omega^2}|, \ldots\}$$

is decreasing not slower than $(\lambda^+)^i$. As all of cylinders $U_{\tau^n\omega^m}$ belong to no more than L_7 Moran covers Z_{l_j}, we get

$$l_j \leq c(\lambda^+)^{L_7^{-1}(j-i)} l_i. \qquad \blacksquare$$

PROPOSITION 5.3: *There exist L_8, L_9 such that*

$$L_8^{-1} l^{-s}\Big(1 + \sum_{l_i \geq L_9^{-1} l} l_i^s A_{l_i}\Big) \leq \tilde{A}_l \leq L_8 l^{-s}\Big(1 + \sum_{l_i \geq L_9 l} l_i^s A_{l_i}\Big).$$

Proof: We will first construct a mapping ϕ that for any pair of intersecting cylinders from Z_l will give us a pair of intersecting cylinders from Z_{l_i} for some $l_i \geq L_9 l$ (L_9 will be chosen in the future) and these new cylinders will have different first digits in their symbolic expansions. Then we will estimate how many preimages does any one of such pairs have.

Let U_{ω^n} and U_{τ^m} be two intersecting cylinders from Z_l. Let η^k be the greatest beginning common subword of ω^n and κ^m; we can write $\omega^n = \eta^k \iota^{n-k}$ and $\kappa^m = \eta^k \tau^{m-k}$. The sequences ι^{n-k} and τ^{m-k} have different first digits. The triple $(U_{\eta^k}, U_{\omega^n}, U_{\kappa^m})$ is the image of $(U, U_{\iota^{n-k}}, U_{\tau^{m-k}})$ under f_{η^k}, hence (by bounded distortion) $U_{\iota^{n-k}}$ and $U_{\tau^{m-k}}$ have approximately the same diameter. So, even if they do not both belong to one Moran cover Z_{l_i}, one of them belongs to some Z_{l_i} together with one of the close ancestors of the other one. Here, by an **ancestor** of U_{ω^n} we mean any of cylinders $U_{\omega^{n-j}}$; the ancestor is **close** if j is uniformly small. In addition, $l_i \geq L_9 l$ for some L_9.

We can freely assume that $U_{\iota^{n-k}}$ and $U_{\tau^{m-k-j}}$ belong together to Z_{l_i}. We define

$$\phi(\omega^n, \kappa^m) = (\iota^{n-k}, \tau^{m-k-j}).$$

We have thus

$$\tilde{A}_l \leq \sharp Z_l + \sum_{l_i \geq L_9 l} A_{l_i} M(l_i, l),$$

where $M(l_i, l)$ is the maximal number of preimages under ϕ that a given pair of intersecting cylinders from Z_{l_i} can have in $Z_l \times Z_l$.

The preimages of $(U_{\omega^n}, U_{\kappa^m})$ are of the form $(U_{\eta^k\omega^n\iota^{j_1}}, U_{\eta^k\kappa^m\iota^{j_2}})$, where η^k is such that $\lambda^+_{\eta^k} \approx l_i/l$, one of numbers j_1, j_2 is zero and the other is bounded. The word η^k can be chosen in $c_1(l_i/l)^s$ ways by (2.4). The word $\iota^{\max(j_1,j_2)}$ can be chosen in at most c_2 ways. Hence,

$$M(l_i, l) \leq c_1 c_2 (l_i/l)^s,$$

which proves the upper bound.

The lower bound is proved in a similar way. Let U_{ω^n} and U_{κ^m} be two intersecting cylinders from two Moran covers $Z_{l^{(1)}}$ and $Z_{l^{(2)}}$, $l^{(1)}$ and $l^{(2)}$ approximately equal (slightly smaller) to l. The number of such pairs is approximately equal to $\tilde{A}_l$, because some close ancestors of U_{ω^n} and U_{κ^m} are intersecting cylinders in Z_l. We will define the mapping φ in the same way as ϕ was defined, only on a greater domain: all such pairs (ω^n, κ^m). We need only to estimate the number of preimages of the mapping φ from below.

When looking for the preimages under φ of a pair $(\iota^{m_1}, \tau^{m_2})$ ($U_{\iota^{m_1}}$ and $U_{\tau^{m_2}}$ being a pair of intersecting cylinders in Z_{l_ι} with different first digits of their symbolic expansions), we will just add a word η^k at their beginning. The word η^k must be chosen such that $f'_{\eta^k} \approx l/l_\iota$. The resulting pair of cylinders' diameters are approximately equal to l, hence it is a preimage of $(\iota^{m_1}, \tau^{m_2})$ under φ. By (2.4), the word η^k may be chosen in approximately $(l/l_\iota)^{-s}$ ways, hence the number of preimages of any $(\iota^{m_1}, \tau^{m_2})$ is not smaller than $c(l/l_\iota)^{-s}$.

We finish the proof like for the upper bound. ∎

Lemma 5.4: *Let $(Z_l(u))$ be a Moran covering and denote by (T_l) the symbolic expansions of its cylinders. The families $X_l(t) = \{U_{\omega^n}(t)\colon \omega^n \in T_l\}$ are then Moran covers, forming a D-covering with $D \le O(|u-t|^\beta)$.*

Proof: From the bounded distortion (2.1), the size of cylinders $U_{\omega^n}(t)$, $\omega^n \in T_l$, is approximately equal to the derivative of the appropriate mapping f_{ω^n}. From Lemma 4.2 we know that

$$\left|\log \frac{f'_{\omega^n}(x;t)}{f'_{\omega^n}(x;u)}\right| \le c \cdot |t-u|^\beta |\log f'_{\omega^n}(x;u)|.$$

Since (for $\omega^n \in T_l$)

$$f'_{\omega^n}(x;u) \approx l/|U|$$

we get

$$|U_{\omega^n}(t)| \in \left(c_1 l\Big(\frac{l}{|U|}\Big)^{c|t-u|^\beta}, c_2 l\Big(\frac{l}{|U|}\Big)^{-c|t-u|^\beta}\right),$$

so we have obtained a D-covering. ∎

As we see, when we choose a Moran covering for some parameter value u, we choose at the same time D-coverings for all t close to u. We will denote the intersection numbers of these coverings by $\tilde{A}_l(t)$ and $A_l(t)$ (skipping D). We recall that by G_p we denote the set of parameter values t, for which the Hausdorff dimension of the limit set Λ_t is strictly smaller than p. We will now prove a key proposition for the proof of Theorem 1.1.

PROPOSITION 5.5: *If $p < s(u)$, then for sufficiently small r,*

$$G_p \cap B_r(u) \subset \bigcup_{l>0} \bigcap_{l_i<l} \{t\colon A_{l_i}(t) \geq l_i^{p-2s(u)+w(r)}\},$$

where $w(r) \leq O(r^\beta)$.

Proof: Choose r and $t \in G_p \cap B_r(u)$. We will work with the Moran covering that we construct from the D-covering by subdivision, like for inequality (3.2). Denote the intersection numbers for this Moran covering by $\tilde{A}_l^o$ and A_l^o. By Proposition 3.2

$$\lim_{l \searrow 0} \frac{\log \tilde{A}_l^o(t)}{-\log l} \geq 2s(t) - p.$$

Denote the left hand side of this inequality by $z(t)$. We claim first that for any small ε there exists $n(\varepsilon)$ such that

$$A_{l_i}^o(t) \geq l_i^{-z(t)+\varepsilon} \tag{5.1}$$

for all $i > n(\varepsilon)$. Here $n(\varepsilon)$ may depend on t. By (3.2) we get

$$A_l(t) \geq L_4^{-2} L_3^{-4s(t)} \left(\frac{l}{|U|}\right)^{4s(t)D(t)/(1+D(t))} A^o_{l^{1/(1+D(t))}|U|^{D/(1+D(t))}}.$$

Substituting (5.1) we get (for l small enough)

$$A_l(t) \geq L_4^{-2} L_3^{-4s(t)} |U|^{(-4s(t)+(\varepsilon-z(t))/(1+D(t)))D(t)} l^{4s(t)D(t)+(\varepsilon-z(t))/(1+D(t))}.$$

By Lemma 5.4 we have

$$D(t) \leq c_1|t-u|^\beta.$$

By Corollary 4.3 we have

$$|s(t) - s(u)| \leq c_2|t-u|^\beta.$$

We get

$$A_l \geq c l^{p-2s(u)+O(r^\beta)}$$

and, as l^{-r^β} will be greater than any constant for small enough l, the assertion follows.

Now we prove the claim. Let us assume (5.1) is not true for some t, that is,

$$A_{l_i}^o(t) < l_i^{-z(t)+\varepsilon}$$

for infinitely many i. Let us choose one such i, very big. By Lemma 5.1 we have

$$A^o_{l_j}(t) < l_j^{-z(t)+\varepsilon/2} \tag{5.2}$$

for all $j > i$ such that $L_4^4 l_j^{-2s(t)+z(t)-\varepsilon/2} < l_i^{-2s(t)+z(t)-\varepsilon}$. Assuming i was big, this is satisfied when $l_j > l_i^{1+\varepsilon_1}$ for some ε_1 that does not depend on i. Denote the smallest such l_j by $\tilde{l}$.

By the definition of $z(t)$, for any ε_2 we can find such a small $l(\varepsilon_2)$ that

$$l_i^{-z(t)+\varepsilon_2} < \tilde{A}^o_{l_i} < l_i^{-z(t)-\varepsilon_2}$$

for all $l_i < l(\varepsilon_2)$.

Let us take very small ε_2 and assume that $l_i < l(\varepsilon_2)$. We have

$$\tilde{A}^o_{l_i}(t) < l_i^{-z(t)-\varepsilon_2}$$

and

$$\tilde{A}^o_{\tilde{l}}(t) > \tilde{l}^{-z(t)+\varepsilon_2} \tag{5.3}$$

(because $\tilde{l}$ must be smaller than $l(\varepsilon_2)$, too).

By Proposition 5.3 we have

$$\tilde{A}^o_{\tilde{l}}(t) \le L_8^2(\tilde{l}/l)^{-s(t)}\tilde{A}^o_{l_i}(t) + L_8\tilde{l}^{-s(t)} \sum_{L_9^{-1}l_i > l_k > L_9\tilde{l}} l_k^{s(t)} A^o_{l_k}(t).$$

By Lemma 5.2 and inequality (5.2) (satisfied by all the l_k) the sum on the right hand side may be estimated by a geometric series with ratio $\gamma^{z(t)-s(t)-\varepsilon/2}$. We get

$$\tilde{A}^o_{\tilde{l}}(t) \le L_8^2\tilde{l}^{(-\varepsilon_1 s(t)-z(t)-\varepsilon_2)/(1+\varepsilon_1)} + L_8\frac{L^{-z(t)+s(t)+\varepsilon/2}}{1-\gamma^{z(t)-s(t)-\varepsilon/2}}\tilde{l}^{-z(t)+\varepsilon/2},$$

which contradicts (5.3) for ε_2 sufficiently small. This contradiction ends the proof of (5.1). We are done. ∎

We will need the following cominatorial lemma.

LEMMA 5.6: *Let H be a regular family of sets E_i with diameters d_i. Given non-negative z, the set E of points that do belong to at least b of the sets E_i can be covered by such a family of open balls $\tilde{E}_i$ with diameters $\tilde{d}_i$ that*

$$\sum \tilde{d}_i^z \le 4^z \cdot \frac{1}{b}\sum d_i^z$$

and $\sup(\tilde{d}_i) \le 4 \cdot \sup(d_i)$.

Proof: The proof is based upon a construction, used in the proof of the Vitali covering theorem. Let us choose from the family H the set of greatest diameter, say E_1. Then we choose the set of greatest diameter among those not intersecting E_1; then once again we choose the greatest set that does not intersect all the sets chosen before, and so on. In that way we will choose some maximal family of disjoint sets from H; let us call it H_1. Next, we will choose (in the same way) the family H_2 as the maximal family of disjoint sets from $H \setminus H_1$, then the family H_3 as the maximal family of disjoint sets from $H \setminus (H_1 \cup H_2)$ and so on up to H_b.

For the set E_i of diameter d_i, denote by $T(E_i)$ a ball of diameter $4d_i$, containing $B_{d_i}(E_i)$. Denote $I_i = T(H_i)$.

We will first prove that for any $i \le b$,

$$E \subset \bigcup_{A \in I_i} A.$$

To prove this, first note that if $x \in E$, then $x \in E_j$ for some $E_j \in H \setminus \bigcup_{n=1}^{i-1} H_n$. It is so because x belongs to at least b intersecting sets from H, and any of families H_n can contain at most one of them.

If $E_j \in H_i$, then $x \in E_j \subset T(E_j) \in I_i$ and we are done. If not, then (from the construction of H_i) E_j must intersect some set $E_m \in H_i$ with greater than or equal diameter. In this case, $x \in E_j \subset B_{d_j}(E_m) \subset B_{d_m}(E_m) \subset T(E_m) \in I_i$ and we are done again.

We have then b families of balls, each one covering E. We have

$$\sum_{i=1}^{b} \sum_{A \in I_i} |A|^z \le 4^z \sum_{i=1}^{b} \sum_{E_j \in H_i} d_j^z \le 4^z \sum_{E_j \in H} d_j^z.$$

Hence,

$$\sum_{A \in I_i} |A|^z \le \frac{1}{b} 4^z \sum_{E_j \in H} d_j^z$$

for some i. ∎

We now give the proof of Theorem 1.1.

Proof: We are going to use Proposition 5.5. Let r be small. Denote $F_i = \{t\colon A_{l_i}(t) \ge l_i^{p-2s(u)+w(r)}\} \cap B_r(u)$, $w(r)$ from Proposition 5.5. Each of these sets is precisely the set of points t in the neighbourhood of u that belong to at least

$l_i^{p-2s(u)+w(r)}$ of sets $I(\omega^n, \kappa^m)$, where $U_{\omega^n}(u)$ and $U_{\kappa^m}(u)$ belong to $Z_{l_i}(u)$ and $\omega_1 \neq \kappa_1$.

By Proposition 4.6 the sets $I(\omega^n, \kappa^m) \cap B_r(u)$ are unions of no more than $Y(r)$ sets of diameter smaller than $Y(r)l_i^{1-v(r)}$. The family Z_{l_i} has no more than $L_4 l_i^{-s(u)}$ elements. Hence by Lemma 5.6 the set F_i can be covered by $L_4 Y(r) l_i^{-p-w(r)}$ balls of diameter $4Y(r)l_i^{1-v(r)}$.

By Proposition 5.5
$$G_p \cap B_r(u) \subset \bigcup_i E_i,$$
where $E_i = \bigcap_{j\geq i} F_j$. We can estimate the upper Minkowski (box) dimension of each of the sets E_i. Covering it with balls of diameter $Y(r)l_i^{1-v(r)}, Y(r)l_{i+1}^{1-v(r)}, \ldots$ we get
$$\overline{\dim}_M(E_i) \leq \frac{p+w(r)}{1-v(r)}.$$

The packing dimension of each E_i (hence of their union as well) cannot be greater than their upper box dimension. Both $w(r)$ and $v(r)$ tend to 0 when r is decreasing. We are done. ∎

COROLLARY 5.7: *If $p < s(u)$ then the set G_p is of first category in the neighbourhood of u.*

Proof: Every set of packing dimension z is a union of a countable family of sets of upper box dimension not greater than z, see [M]. The upper box dimension of a set and of its closure is the same. If the upper box dimension of a set is smaller than the dimension of the space, the set is thus nowhere dense. Hence, every set of packing dimension smaller than the dimension of the space is contained in a countable union of nowhere dense closed sets. ∎

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